AP Calculus AB
L'Hopital's Rule

Name Key

Suppose f and g are differentiable and $g'(x) \neq 0$. We want to determine the $\lim_{x \to a} \frac{f(x)}{g(x)}$.

If we have an indeterminate form (either $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

L'Hopital's Rule states that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the limit is an indeterminate form.

Examples:

1. $\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0}$

$\lim_{x \to 0} \frac{\cos x}{1} = 1$

2. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{0}{0}$

$\lim_{x \to 2} \frac{2x}{1} = 4$

3. $\lim_{x \to 2} \frac{x^2 - 4}{x + 1} = \frac{0}{3} = 0$

(not l'hopitals)

4. $\lim_{x \to 0} \frac{e^x - 1}{x^3} = \frac{0}{0}$

$\lim_{x \to 0} \frac{e^x}{3x^2} = \frac{1}{0}$

(not l'hopitals)

$\lim_{x \to 0^-} \frac{e^x}{3x^2} \quad \frac{(+)}{(+)} \quad \infty$

$\lim_{x \to 0^+} \frac{e^x}{3x^2} \quad \frac{(+)}{(+)} \quad \infty$

$= \infty$

5. $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{0}{0}$

$\lim_{x \to 0} \frac{\sin x}{2x} = \frac{0}{0}$

$\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$

6. $\lim_{x \to 0} \frac{e^x - 1}{x} = \frac{0}{0}$

$\lim_{x \to 0} \frac{e^x}{1} = 1$

7. $\lim_{x \to 2} \frac{\ln(x - 1)}{x - 2} = \frac{0}{0}$

$\lim_{x \to 2} \frac{\frac{1}{x-1}}{1} = 1$

8. $\lim_{x \to 0} \frac{e^x - 1}{\sin 2x} = \frac{0}{0}$

$\lim_{x \to 0} \frac{e^x}{2\cos 2x} = \frac{1}{2}$

9. $\lim_{x \to \infty} \frac{x^2}{2x^3 + x + 5} = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{2x}{6x^2 + 1} = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{2}{12x} = 0$

10. $\lim_{x \to -\infty} \frac{x^4 + 5}{5x^4 + 8x^3} = \frac{\infty}{\infty}$

$\lim_{x \to -\infty} \frac{4x^3}{20x^3 + 24x^2} = \frac{-\infty}{-\infty}$

$\lim_{x \to -\infty} \frac{12x^2}{60x^2 + 48x} = \frac{\infty}{\infty}$

$\lim_{x \to -\infty} \frac{24x}{120x + 48} = \frac{-\infty}{-\infty}$

$\lim_{x \to -\infty} \frac{24}{120} = \frac{1}{5}$

11. $\lim_{x \to \infty} \frac{3x^2}{e^x} = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{6x}{e^x} = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{6}{e^x} = 0$

12. $\lim_{x \to +\infty} \frac{e^x}{x^3} = \frac{\infty}{+\infty}$

$\lim_{x \to \infty} \frac{e^x}{3x^2} = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{e^x}{6x} = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{e^x}{6} = \frac{\infty}{6} = \infty$

Other indeterminate forms:

$0 * \infty$ $\qquad 0^0$ $\qquad 1^\infty$ $\qquad \infty^0$ $\qquad \infty - \infty$

When in these forms, we try to re-write the limit as a fraction which yields the indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ so that we may use L'Hopital's Rule.

Examples:

1. $\lim_{x\to\infty} 2xe^{-3x} = \infty \cdot 0$

$$\lim_{x\to\infty} \frac{2x}{e^{3x}} = \frac{\infty}{\infty} \qquad \lim_{x\to\infty} \frac{2}{3e^{3x}} = 0$$

2. $\lim_{x\to 0^+} x\ln x = 0 \cdot -\infty$

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{\infty} \qquad \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} \qquad \lim_{x\to 0^+} -x = 0$$

3. $\lim_{x\to\infty} x\sin\frac{\pi}{x} = \infty \cdot 0$

$$\lim_{x\to\infty} \frac{\sin\frac{\pi}{x}}{\frac{1}{x}} = \frac{0}{0}$$

$$\lim_{x\to\infty} \frac{\cos\frac{\pi}{x}(+\pi \cancel{x^{-2}})}{+\cancel{x^{-2}}} \qquad \lim_{x\to\infty} \pi\cos\frac{\pi}{x} = \pi$$

4. $\lim_{x\to\frac{\pi}{4}} (1-\tan x)\sec 2x = 0 \cdot \frac{1}{0}$

$$\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\cos 2x} = \frac{0}{0}$$

$$\lim_{x\to\frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1$$

5. $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \frac{1}{0} - \frac{1}{0}$

$$\lim_{x\to 0} \frac{x-\sin x}{x\sin x} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{1-\cos x}{(1)\sin x + x\cos x} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{\sin x}{\cos x + (1)\cos x - x\sin x} \qquad \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

Note: The following forms are determinate:

$\infty + \infty \to \infty$ $\qquad -\infty - \infty \to -\infty$ $\qquad 0^\infty \to 0$ $\qquad 0^{-\infty} \to \infty$